A CONVERGING SHOCK-WAVE IN A GAS OF VARIABLE DENSITY

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The problem of a spherical (and cylindrical) shock-wave focussing at the center (axis) of symmetry in a homogeneous quiescent gas was investigated by Guderley [1]. He constructed a self-similar solution describing the gas motion for a strong converging shock-wave whose front propagates as a power of time. Below, the problem is considered of a converging shock-wave in a gas of variable density.

1. Let the initial (undisturbed) condition of an ideal gas be given by the formulas

$$p \equiv p_0, \qquad \rho = \omega r^s, \qquad v \equiv 0, \qquad (s \geqslant 0) \tag{1.1}$$

where r, p, ρ, v are, respectively, the distance of a particle from the center (axis, or plane) of symmetry, the pressure, density, and mass velocity.

The equations of one-dimensional adiabatic motion of an ideal gas have the form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial r} + (\nu - 1)\frac{\rho v}{r} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho}\right) + v\frac{\partial}{\partial r}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{1.2}$$

where t is the time, γ the adiabatic index; $\nu = 1$ for plane, $\nu = 2$ for cylindrical and $\nu = 3$ for spherical waves.

The conditions at the shock-wave may be taken in the form

$$v_2 = v_1 + \frac{2}{\gamma+1}c_1\left(\frac{1}{M} - M\right), \qquad p_2 = p_1\left[1 + \frac{2\gamma}{\gamma+1}(M^2 - 1)\right]$$

$$\rho_2 = \rho_1\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \qquad \left(M = \frac{v_1 - D}{c_1}\right) \tag{1.3}$$

where c is the speed of sound and D the speed of translation of the shock-wave front. Index 1 corresponds to conditions ahead of the shock, and index 2 behind the shock, where necessarily $|M| \geqslant 1$.

Let the position of the converging shock-wave be determined by a power law: $r(t) = a(-t)^{\delta}$, where $\delta > 0$, $a > 0$ and conditions ahead of the shock are given by Equations (1.1). Then from (1.1), (1.3) we have

$$c_1^2 = \frac{\gamma p_0}{\omega r^s}, \qquad M = \frac{\delta a^{\Delta} \sqrt{\omega}}{\sqrt{\gamma p_0}} r^{1-\Delta+{}^1\!/_2 s} \qquad \left(\Delta = \frac{1}{\delta}\right) \tag{1.4}$$

In order that the inequality $M \geqslant 1$ be realized as $r \to 0$, it is necessary that $\delta < 2/(2 + s)$. If $\delta \leqslant 2/(2 + s)$, then $M \to \infty$ as $r \to 0$; on the other hand if $\delta = 2/(2 + s)$, then M is a constant (which must be no less than 1), and the wave has constant intensity.

We will suppose the motion behind the shock to be self-similar [2] with determining parameters r, t, ω, a. With $\delta = 2/(2 + s)$ the dimensions of p_0 are expressed in terms of the dimensions of a and ω, which is evident from (1.4), and the assumption of self-similarity is consistent with the conditions (1.1), (1.3). However, if $\delta < 2/(2 + s)$ then since the shock-wave becomes strong near $r = 0$, the initial pressure p_0 may be neglected, and conditions (1.3) taken in the form

$$v_2 = -\frac{2\delta}{\gamma+1} a^{\Delta} r^{1-\Delta}, \quad p_2 = \frac{2\delta^2 a^{2\Delta}\omega}{\gamma+1} r^{2-2\Delta+s}, \quad \rho_2 = \frac{\gamma+1}{\gamma-1} \omega r^s \tag{1.5}$$

The dependent variables v, p, ρ are expressed, under the assumption of self-similarity, by the formulas

$$v = \frac{\delta r}{t} V(\lambda), \qquad p = \frac{\delta^2 r^2}{t^2} \omega r^s P(\lambda), \qquad \rho = \omega r^s R(\lambda) \qquad \left(\lambda = \frac{r}{a\,|\,t\,|^{\delta}}\right) \tag{1.6}$$

The notation for the multipliers δ and δ^2 of V and P differs from that used in [2].

Substituting Equations (1.6) into Equations (1.2), and introducing in place of P a new dependent variable $z = \gamma P/R$, we obtain the known ordinary differential equations for self-similar motion

$$\frac{dz}{dV} = \frac{z}{V-1} \frac{(2+\varkappa-\varkappa\gamma-2V\delta)A + (\gamma-1)\delta(V-1)B}{(\varkappa-\nu V\delta)A + \delta(1-V)B} \tag{1.7}$$

$$\frac{d\ln\lambda}{dV} = \frac{\delta A}{(\varkappa-\nu V\delta)A + \delta(1-V)B} \tag{1.8}$$

$$(V-1)\frac{d\ln R}{d\ln\lambda} = -(s+\nu)V - \frac{(\varkappa-\nu V\delta)A + \delta(1-V)B}{\delta A} \tag{1.9}$$

Here

$$A(V, z) = z - (1-V)^2 \tag{1.10}$$

$$B(V) = (\nu-1)V^2 + \frac{1-\nu\delta-\varkappa}{\delta} V + \frac{\varkappa}{\delta}, \qquad \varkappa = \frac{2-\delta(s+2)}{\gamma} \geqslant 0$$

Initial conditions for this system are obtained from the conditions at the shock-wave, in which it is necessary to substitute the expressions (1.6). For $\delta = 2/(2+s)$ the initial conditions have the form ($M \geqslant 1$)

$$V = \frac{2}{\gamma+1}\left(1 - \frac{1}{M^2}\right), \qquad R = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}$$
$$z = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}\left\{\left(1 - \frac{\gamma-1}{2\gamma M^2}\right)\left[1 + \frac{2}{(\gamma-1)M^2}\right]\right\} \quad \text{for } \lambda = 1 \tag{1.11}$$

In the case $\delta < 2/(2+s)$ we obtain the following initial conditions:

$$V = \frac{2}{\gamma+1}, \qquad z = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}, \qquad R = \frac{\gamma+1}{\gamma-1} \quad \text{for } \lambda = 1 \tag{1.12}$$

If the number δ is known, the problem reduces to integration of one differential equation (1.7) for given initial conditions, and two quadratures; Equation (1.9) may be replaced by the integral of adiabaticity [2].

For the determination of δ we require that the solution constructed have physical sense and describe the motion of gas with a converging shock-wave all the way to its arrival at the center (axis or plane) of symmetry, that is, for all $t \leqslant 0$. We suppose the gas motion to be continuous in the region behind the shock.

In order that v and p be bounded for $t = 0$, $r \neq 0$, it is necessary, as is evident from (1.6), that $V = 0$, $P = 0$, and consequently also $z = 0$ at $\lambda = \infty$. Therefore, the integral curve of Equation (1.7) in the Vz-plane, giving the solution of the problem, should pass through the origin of coordinates O, which is a singular point (node) of Equation (1.7). One integral curve passes through it with slope $(dz/dV)_0 = 1/\kappa$, and the others are tangent to the V-axis.

The initial points in the Vz-plane, determined by (1.11) lie on the parabola

$$z = (1 - V)\left(1 + \frac{\gamma-1}{2}V\right) \tag{1.13}$$

where as M increases from 1 to ∞ the coordinates V, z of the initial point change correspondingly:

$$\text{from } 0 \text{ to } V^\circ = \frac{2}{\gamma+1}, \qquad \text{from } 1 \text{ to } z^\circ = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}$$

Thus for $\delta = 2/(2+s)$ the solution passes through points of the parabola (1.13) between (0, 1) and $N(V^\circ, z^\circ)$, whereas for $\delta < 2/(2+s)$ it passes through the point N.

This segment of the parabola and the point O lie on opposite sides of the curve $A(V, z) = 0$, so that the desired integral curve must intersect it. In moving along the integral curve which gives the solution from the initial point to the point O, the value of λ should increase monotonically from 1 to ∞; otherwise one and the same λ would correspond to different V and z, and the flow would be discontinuous. But for $A = 0$, as is evident from (1.8), λ has generally speaking an extremum. Therefore, in order that the solution have physical meaning, it is necessary that the desired integral curve intersect the parabola $A = 0$ in a singular point of the differential equation, at which $A = 0$ and $V = 1$, or $A = B = 0$.

The first possibility corresponds to the point $Q(1, 0)$. A detailed analysis of the system (1.7)-(1.9) in the neighborhood of the singular point Q shows that none of the curves passing through it can be the desired solution.

The second possibility corresponds to the pair of singular points $P_{1,2}$, whose z-coordinates are determined by the equation $z_{1,2} = (V_{1,2} - 1)^2$, and V_1 and V_2 ($V_1 \leqslant V_2$) are the roots of the quadratic equation $B(V) = 0$, where $B(V)$ is determined by Equation (1.10).

Since $\kappa \geqslant 0$, V_1 and V_2 have the same sign. Let $V_1 \leqslant V < 0$; then the desired integral curve, in going from the initial point, must intersect the axis $V = 0$ in order to pass through P_1 or P_2. In so doing it must remain in the region $A > 0$, where the initial point lies. But from (1.8) with $A > 0$, $V = 0$ (therefore $B \geqslant 0$) it follows that $d \ln \lambda / dV > 0$. Therefore, as the curve passes from $V > 0$ to $V < 0$, λ decreases, which contradicts the requirement of monotonic increase of λ along the curve from the initial point to point O.

Consequently, a necessary condition for possible construction of a solution with the required properties is the existence of a non-negative root of the quadratic equation $B(V) = 0$. Substituting into it κ from Equation (1.10), we obtain after a simple transformation

$$\frac{1}{\delta} - \frac{2+s}{2} = \frac{\gamma(\nu-1)V[V-1+s/2(\nu-1)]}{(2-\gamma)V-2}$$

Hence it follows that the two requirements $\delta \leqslant 2/(2+s)$ and $V \geqslant 0$ are compatible only for $0 \leqslant V \leqslant 1 - s/2(\nu - 1)$, for which it is necessary that $s \leqslant 2(\nu - 1)$. When the last inequality is satisfied, δ may lie in the range

$$0 \leqslant \frac{1}{\delta} - \frac{2+s}{2} \leqslant \frac{\gamma(\nu-1)}{2} V_0^2 \tag{1.14}$$

where

$$V_0 = \frac{2}{2-\gamma}\left\{1 - \sqrt{1 - \frac{2-\gamma}{2}\left[1 - \frac{s}{2(\nu-1)}\right]}\right\}$$

and each δ, except the minimum, corresponds to two values of V. With increasing δ the larger of these (V_2) increases from V_0 to $1 - s/2(\nu - 1)$, and the smaller (V_1) decreases from V_0 to 0.

Thus if $s \leqslant 2(\nu - 1)$ and δ is subject to condition (1.14), then the necessary conditions for construction of a solution are satisfied. It is then possible, generally speaking, to construct self-similar solutions of two types (with γ, ν and s fixed).

1. We suppose that $\delta < 2/(2 + s)$; the desired integral curve should pass through the fixed initial point N and run into point O, intersecting the parabola $A(V, z) = 0$ in one of the singular points $P_{1,2}$. The condition for determining δ is that the point N lie exactly on the integral curve passing through the singular point in the specified direction. The choice of δ and construction of the solution are realized by the method of successive approximations: Equation (1.7) is integrated numerically for different δ (lying within the limits (1.14)) from the singular point, until N lies on the curve.

2. Let $\delta = 2/(2 + s)$; the field of integral curves is then fixed, and any point on the segment (1.13) of the parabola may serve as initial point. The singular point P_1 has coordinates (0, 1), and analysis shows that for any $s \geqslant 0$, along any curve passing through this point, λ decreases as P_1 is approached from the region $V \geqslant 0$, where the initial point lies. The segment of the axis from O to P_1 gives a trivial solution corresponding to quiescence.

Thus the desired solution for $\delta = 2/(2 + s)$ can go only through point P_2. If the integral curve of Equation (1.7) passing through O and P_2 (with variation of λ in the desired direction) intersects the initial parabola (1.13) with $0 \leqslant V \geqslant V^\circ$, then it gives the solution of the problem. The number M is determined from Equation (1.11) using the resulting point of intersection.

The first case corresponds to a strong shock-wave, and the second to a wave of definite constant (moderate) intensity. We note that points $P_{1,2}$ for solutions of the first type lie between points $P_{1,2}$ for the second case (for the same γ, ν and s).

With $s = 0$ ($\nu \neq 1$) no solution of the second type exists, since for $s = 0$, $\delta = 1$ point P_2 coincides with $Q(1, 0)$. Solutions of the first type (with a strong wave) were constructed in [1]; the integral curve passes through the singular point P_2, which is a saddle.

However, if the difference $s - 2(\nu - 1)$ is negative and sufficiently small ($\nu \neq 1$), then conversely only a solution of the second type can be constructed, with a moderate wave. In this case point P_2 is a saddle; of the two curves passing through it, along only one does λ increase in passing from the region $A > 0$ to $A < 0$.

The slope dz/dV of this curve becomes arbitrarily large as $[s - 2(\nu - 1)] \to - 0$, and the point P_2 itself is arbitrarily near to $(0, 1)$; this also guarantees the intersection of the curve with the segment of the initial parabola. The number M, corresponding to the point of intersection, tends to 1 as $[s - 2(\nu - 1)] \to - 0$. It is impossible to construct a solution with a strong wave here; the integral curve passing through P_1 and P_2 (lying near to point $(0, 1)$), does not reach point N.

As s increases from 0 to $2(\nu - 1)$ the type of solution apparently changes in the following way. At first (for s close to 0) only a solution with a strong shock wave is possible, that is with $\delta < 2/(2 + s)$, and the rate of approach of M to ∞ as $r \to 0$ decreases as s increases. For a certain s there is a solution passing through point N such that $\delta = 2/(2 + s)$. This is a case of a solution that is intermediate between the first and second types, where the wave has constant but infinitely great intensity. With further increase in s only the solution of the second type (with a moderate wave) is realized, and the constant M decreases from ∞ to 1 as s increases to $2(\nu - 1)$.

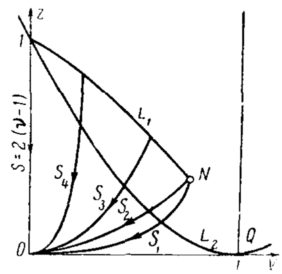

Fig. 1.

A solution of the first type may, generally speaking, be constructed also for $s < 0$; here a solution of the second type is impossible, since for $\delta = 2/(2 + s)$ the singular point P_2 lies in the region $V > 1$, separated from the initial parabola by the straight line $V = 1$ (an integral of Equation (1.7)).

A schematic view of the solutions for various s is given in Fig. 1. Here L_1 is the parabola of initial points, L_2 the curve $z = (1 - V)^2$ on which lie the singular points $P_{1,2}$, arrows indicate the direction of increasing λ, and $s_i < s_j$ for $i < j$ and s_i does not exceed $2(\nu - 1)$. The value s_1 corresponds to a solution of the first type, s_2 to the dividing case, and s_3 and s_4 to solutions of the second type, where $M_3 > M_4$. For $\nu \neq 1$ the value $s = 2(\nu - 1)$ corresponds only to a trivial solution – a segment of the z-axis.

In the case of a plane wave ($\nu = 1$) there is only one singular point P_1, the coordinate V_1 of which is related to δ by

$$\frac{1}{\delta} - \frac{2+s}{2} = \frac{\gamma V_1 s}{2[(2-\gamma)V_1 - 2]} \qquad (1.15)$$

A solution of the second type is impossible for $s \neq 0$, since with $\delta = 2/(2 + s)$ we obtain $V_1 = 0$, and point P_1 has coordinates (0, 1). For $s = 0$, as is evident from (1.15), δ should be equal to 1. In this case $B(V) \equiv 0$, and passage of the solution through the parabola $A = 0$ is possible, because λ has no extremal on it. Equations (1.7)-(1.9) have the simple solution

$$V = V(1)\lambda^{-1}, \qquad z = z(1)\lambda^{-2}, \qquad R = R(1)$$

Here $V(1)$, $z(1)$, $R(1)$ are given by the initial conditions (1.11).

Thus for $s < 2(\nu - 1)$ a self-similar solution may be constructed with a strong or moderate converging shock-wave. After construction of the solution all characteristics of the solution are known for $t = 0$, and its distribution is self-similar. Consequently, a Cauchy problem may be solved for the motion of the gas when $t \geqslant 0$ in the same self-similar variables. In the case $s = 0$ such a solution is constructed in [1] with a reflected diverging shock-wave.

2. For $s \geqslant 2(\nu - 1)$ the particular points $P_{1,2}$ for the solutions of the two types lie in the region $V \leqslant 0$. Consequently, the necessary condition for construction of automodel solutions with strong or moderate (M = const 1) converging shock-waves is infringed.

Uniqueness of the self-similar solution is here trivial, in the absence of perturbations. Since for $[s - 2(\nu - 1)] \to -0$ the intensity of the shock-wave for a self-similar solution of the second type decreases, tending to 0, it is natural to suppose that for $s \geqslant 2(\nu - 1)$ there exist motions with converging shock-waves, that are weak near $r = 0$.

Having this in mind, for the case $s \geqslant 2(\nu - 1)$ we linearize Equations (1.2). We set

$$p = p_0 + p^\circ, \qquad \rho = \omega r^s + \rho^\circ$$

and will suppose

$$|p^\circ| \ll p_0, \quad |\rho^\circ| \ll \omega r^s, \quad |v| \ll c_1 = \sqrt{\gamma p_0 / \omega r^s}.$$

We write the linearized equations of motion

$$\omega r^s \frac{\partial v}{\partial t} + \frac{\partial p^\circ}{\partial r} = 0, \qquad \frac{\partial \rho^\circ}{\partial t} + \omega r^s \left(\frac{\partial v}{\partial r} + \frac{\nu + s - 1}{r} v\right) = 0$$

$$\frac{1}{p_0}\frac{\partial p^\circ}{\partial t} - \frac{\gamma}{\omega r^s}\frac{\partial \rho^\circ}{\partial t} - \frac{\gamma s v}{r} = 0 \qquad (2.1)$$

We denote by $u(r, t)$ the displacement of a particle from its initial (undisturbed) position; then $v = \partial u/\partial t$. Here, as usual in linearized theory, the difference between the Eulerian and the Lagrangean coordinate of a particle is neglected. Integrating the second and third of Equations (2.1) with respect to t under the conditions that $u = 0$, $p^\circ = 0$, $\rho^\circ = 0$ as $t \to -\infty$, we obtain after a simple transformation

$$p^\circ = -\gamma p_0\left(\frac{\partial u}{\partial r} + \frac{\nu - 1}{r} u\right), \qquad \rho^\circ = \frac{\omega r^s}{\gamma p_0} p^\circ - s\omega r^{s-1} u \tag{2.2}$$

Substituting (2.2) into the first of Equations (2.1), we obtain an equation for the basic dependent variable $u(r, t)$:

$$\frac{\omega r^s}{\gamma p_0}\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r} + (\nu - 1)\frac{u}{r}\right] \tag{2.3}$$

The characteristics of this linear hyperbolic second-order equation are

$$t + \frac{2\sqrt{\omega}\, r^{1+{}^1/_2 s}}{(2+s)\sqrt{\gamma p_0}} = \xi, \qquad t - \frac{2\sqrt{\omega}\, r^{1+{}^1/_2 s}}{(2+s)\sqrt{\gamma p_0}} = \eta$$

In characteristic variables, after changing the dependent variable, we obtain an Euler-Poisson-Darboux equation:

$$\frac{\partial^2 \varphi}{\partial\xi\partial\eta} - \frac{m}{\xi - \eta}\left(\frac{\partial\varphi}{\partial\xi} - \frac{\partial\varphi}{\partial\eta}\right) = 0 \qquad \left(\varphi = u r^{\nu-1},\ m = \frac{1}{2} - \frac{\nu}{2+s}\right) \tag{2.4}$$

It is known that the general solution of Equation (2.4) can be found in closed form if m is an integer. Thus with $s = 2(\nu - 1)$ we have $m = 0$, and the general solution of Equation (2.3) has the form

$$u = r^{1-\nu}\left[f\left(t + \sqrt{\frac{\omega}{\gamma p_0}}\frac{r^\nu}{\nu}\right) + g\left(t - \sqrt{\frac{\omega}{\gamma p_0}}\frac{r^\nu}{\nu}\right)\right]$$

For $\nu = 1$ it becomes the known solution for a plane wave.

If the displacement $u(r, t)$ is found, the excess pressure and density are determined by Equation (2.2) and the velocity by $v = \partial u/\partial t$. We can, however, obtain also a separate equation for this variable:

$$\frac{\omega r^s}{\gamma p_0}\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial r}\left[\frac{\partial v}{\partial r} + (\nu - 1)\frac{v}{r}\right], \qquad \frac{\omega r^s}{\gamma p_0}\frac{\partial^2 p^\circ}{\partial t^2} = \frac{\partial^2 p^\circ}{\partial r^2} + \frac{\nu - s - 1}{r}\frac{\partial p^\circ}{\partial r} \tag{2.5}$$

It is easy to see that Equation (2.3) has a group-invariant solution of the form

$$u = C r^{1+k} w(x), \qquad x = \frac{1}{2}\left(\frac{2+s}{2}\sqrt{\frac{\gamma p_0}{\omega}}\frac{t}{r^{1+{}^1/_2 s}} + 1\right) \tag{2.6}$$

weak shock.

At $r = 0$ we have the boundary condition $v(0, t) = 0$, expressing the absence of sources or sinks.

In Fig. 2 the curves S_1 and S_2 represent characteristics, the incoming and reflected wave-fronts, whose equations are $x = 0$, $x = 1$ (cf. (2.6)), and 1, 2 and 3 are regions in the r-t-plane in which solutions must be found.

From Equations (2.7) and (2.8) it follows that with the condition of continuity of displacement on the incoming and reflected front, the relation (2.12) is satisfied on it automatically.

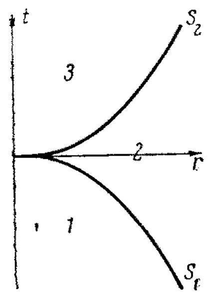

Fig. 2.

Before the arrival of the perturbation, for $-\infty < x \leqslant 0$ (region 1 in Fig. 2) we have

$$w_1 = q_1 = \sigma_1 \equiv 0$$

The displacement on the front of the converging wave (at $x = 0$) changes continuously, and the velocity and pressure undergo a finite jump. Without loss of generality (because of the linearity of the equations) we require

$$w_2(0) = 0, \qquad w_2'(0) = 1 \tag{2.13}$$

For the determination of the motion in region 2 (for $0 \leqslant x \leqslant 1$) it is necessary to construct a solution of the hypergeometric equation (2.9) with parameters (2.10) for the initial conditions (2.13), the quantity k remaining undetermined for the present. The point $x = 0$ is singular for the hypergeometric equation, and consideration of its linearly-independent solutions in the neighborhood of this point [3] for different θ shows that both conditions (2.13) can be satisfied only for $\theta = 0$. Thus

$$k = \frac{2+s}{4} - \frac{\nu}{2} = \frac{2+s}{2} m \tag{2.14}$$

It is possible to consider also solutions with $\theta \neq 0$; then the pressure and velocity on the front of a converging wave will approach 0 or ∞.

For $\theta = 0$ the solution of Equation (2.9) satisfying conditions (2.13) is unique, and has the form (considering the values of α, β, k)

$$w_2 = xF(1-m, m; 2; x) \tag{2.15}$$

where F is a hypergeometric series, converging absolutely for $|x| \leqslant 1$.

where w is a twice-differentiable dimensionless function of the dimensionless argument x, and C is an arbitrary constant of dimensions $(\mathrm{cm})^{-k}$.

If the displacement is given by Equation (2.6), then for v and p° we obtain, using $v = \partial u/\partial t$ and (2.2)

$$v = C\,\frac{2+s}{4}\sqrt{\frac{\gamma p_0}{\omega}}\,r^{k-\frac{1}{2}s}q(x), \qquad p^\circ = -\gamma p_0 C\,\frac{2+s}{4}\,r^k\sigma(x) \tag{2.7}$$

where the functions $g(x)$ and $\sigma(x)$ are expressed in terms of $w(x)$ in the following form:

$$q(x) = w'(x), \qquad \sigma(x) = (1-2x)\,w'(x) + \frac{[4(k+\nu)\,w(x)]}{2+s} \tag{2.8}$$

Substituting (2.6) into (2.3) we obtain an ordinary differential equation for the function $w(x)$. This is the hypergeometric equation

$$x(1-x)\,w'' + [\theta - (1+\alpha+\beta)\,x]\,w' - \alpha\beta w = 0 \tag{2.9}$$

with the following values of the parameters α, β, θ:

$$\alpha = -\frac{2k}{2+s}, \qquad \beta = -\frac{2(k+\nu)}{2+s}, \qquad \theta = \frac{1}{2} - \frac{3k+\nu}{2+s} \tag{2.10}$$

After substitution of Expression (2.7) into Equation (2.5) we obtain analogous differential equations for q and σ. They are also hypergeometric, with parameters

$$\begin{aligned} &\alpha = 1 - \frac{2k}{2+s}, \qquad \beta = 1 - \frac{2(k+\nu)}{2+s}, \qquad \theta = \frac{3}{2} - \frac{2k+\nu}{2+s} \qquad \text{for } q \\ &\alpha = -\frac{2k}{2+s}, \qquad \beta = 2 - \frac{2(k+\nu)}{2+s}, \qquad \theta = \frac{3}{2} - \frac{2k+\nu}{2+s} \qquad \text{for } \sigma \end{aligned} \tag{2.11}$$

We construct a solution of the form (2.6) with a weak shock-wave converging in quiescent gas. At $t = 0$ reflection of the shock occurs from the center (axis, or plane) of symmetry.

The conditions on the shock-wave (1.3) may, in the case of a weak shock ($M \to 1$), be taken in the form

$$p_2^\circ - p_1^\circ = D\omega r^s (v_2 - v_1), \qquad D = \pm\sqrt{\gamma p_0/\omega r^s} \tag{2.12}$$

with the plus sign for a diverging, and minus for a converging wave.

The third condition on the shock-wave is satisfied automatically in virtue of the nearness of adiabatic and isentropic compressions for a

The functions $q_2(x)$ and $\sigma_2(x)$ are determined by using (2.8); it is convenient, however, to take advantage of the fact that they satisfy hypergeometric equations with the parameters (2.11). The solutions of these equations with initial conditions $q_2(0) = 1$, $\sigma_2(0) = 1$, resulting from (2.13) and (2.8), have the form

$$q_2 = F(1-m, m; 1; x), \qquad \sigma_2 = F(-m; 1+m; 1; x) \tag{2.16}$$

In order to determine the motion in region 3 (for $1 \leqslant x < \infty$), it is necessary to find a solution of Equation (2.9) for which $v(0, t) = 0$. Of the two linearly-independent solutions of the hypergeometric equation [4] in the vicinity of the singular point $x = \infty$ only one satisfies this condition, namely

$$w_3 = Ex^m F(-m, 1-m; 2-2m; x^{-1}) \tag{2.17}$$

where the series in (2.17) converges absolutely for $|x| \leqslant 1$.

On the front of the reflected wave the displacement changes continuously; therefore $w_2(1) = w_3(1)$, from which is found the constant

$$E = \frac{\Gamma(1-m)}{\Gamma(1+m)\,\Gamma(2-2m)} \tag{2.18}$$

Using the same considerations that led to Equation (2.16), we determine $q_3(x)$ and $\sigma_3(x)$:

$$q_3 = Emx^{m-1}F(1-m, 1-m; 2-2m; x^{-1}) \tag{2.19}$$
$$\sigma_3 = 2E(1-2m)x^m F(-m, -m; -2m; x^{-1})$$

Thus the functions w, q, σ are determined in the entire region, and with them the solution is found for the given problem of reflection of a converging wave. From the equations expressing the solution in regions 2 and 3 it follows that the condition of linearization of the equations, that is, the conditions $|u| \ll r$, $|v| \ll c_1$, $|p^\circ| \ll p_0$, are satisfied as $r \to 0$, $t \to 0$ if $s > 2(\nu - 1)$ and $k > 0$. Violation of these conditions occurs only for x near to 1 (the reflected front) since the series (2.16) and (2.19) diverge for $x = 1$.

We investigate the behavior of the solution on certain characteristic lines in the r-t-plane.

1. On the front of the converging wave ($x = 0$) we have

$$u = 0, \; v = C\frac{2+s}{4}\sqrt{\frac{\gamma p_0}{\omega}}\, r^{k-1/2s}, \qquad p^\circ = -\gamma p_0 C\frac{2+s}{4} r^k$$

Although the velocity on the shock grows without limit as $r \to 0$, the ratios v/c_1 and p°/p_0 are infinitesimally small there for $s > 2(\nu - 1)$. This permits the conclusion that the converging wave is weak near $r = 0$ and justifies the use of the approximate relations (2.12).

2. We find the distribution of quantities at $t = 0$. In order to calculate the values of the hypergeometric series in (2.15) and (2.16) at $x = 1/2$ required for this, we use the equations

$$F\left(\alpha, \beta; \frac{\alpha+\beta+1}{2}; \frac{1}{2}\right) = \frac{\Gamma\left(\frac{\alpha+\beta+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}$$

$$F\left(\alpha+1, -\alpha; 2; \frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{3}{2}+\frac{\alpha}{2}\right)}$$

The first of these equations appears in [5], and the second may be obtained as its limiting case for $(\alpha + \beta + 1) \to 0$. Now determining $w_2(1/2)$, $q_2(1/2)$, $\sigma_2(1/2)$, we find u, v and p° at $t = 0$ from (2.6) and (2.7):

$$u = \frac{C\sqrt{\pi}\, r^{1+k}}{4\Gamma(1 + {}^1/_2 m)\,\Gamma({}^3/_2 - {}^1/_2 m)}$$

$$v = \frac{C(2+s)\sqrt{\pi}\sqrt{\gamma p_0}\, r^{k-s/2}}{4\sqrt{\omega}\,\Gamma({}^1/_2 + {}^1/_2 m)\,\Gamma(1 - {}^1/_2 m)}$$

$$p^\circ = -\frac{\gamma p_0 C(2\nu + 2 + s)\sqrt{\pi}\, r^k}{16\Gamma(1 + {}^1/_2 m)\,\Gamma({}^3/_2 - {}^1/_2 m)}$$

3. On the front of the reflected wave (for $x = 1$) the deflection is finite and equal on both sides of the shock. Calculating $w_2(1)$, we find u on the reflected front from (2.6):

$$u = \frac{C}{\pi}\cos\left(\frac{\pi\nu}{2+s}\right)\frac{4r^{1+k}}{1 - [2\nu/(2+s)]^2}$$

To estimate q and σ near $x = 1$ we use the equation given in [5]:

$$\lim_{x\to 1-0}\frac{F(\alpha, \beta; \alpha+\beta; x)}{\ln(1-x)} = -\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

From this it is evident that the velocity and pressure become logarithmically infinite near the reflected front, where it is seen that the coefficient of the logarithmic term is the same on both sides of the front. We have the equation

$$\lim_{x\to\pm 0}\frac{v}{\ln|1-x|} = -C\frac{2+s}{4\pi}\cos\frac{\pi\nu}{2+s}\sqrt{\frac{\gamma p_0}{\omega}}\,r^{k-1/2 s} \tag{2.20}$$
$$\lim_{x\to\pm 0}\frac{p^\circ}{\ln|1-x|} = -\gamma p_0 C\frac{2+s}{4\pi}\cos\frac{\pi\nu}{2+s}\,r^{k}$$

It is evident that near the reflected front the condition of linearization is violated.

4. Values of u, v, p° near the center (axis, or plane) of symmetry after reflection of the wave are determined by the asymptotic equations

$$u = brt^m, \qquad v = bmrt^{m-1}, \qquad p^\circ = -\gamma p_0 b\nu t^m \qquad \left(b = C\left(\frac{2+s}{4}\sqrt{\frac{\gamma p_0}{\omega}}\right)^m E\right) \tag{2.21}$$

obtained from the solution in region 3 as $r \to 0$ (or $x \to \infty$). If the converging wave is a wave of compression, $C < 0$; it follows from Equation (2.20) that a rarefaction occurs near the reflected front (on both sides of it), but a compression at $r = 0$.

If m, determined by (2.4), is an integer, the general solution of Equation (2.3) is found in closed form. Then all the parameters of the hypergeometric series (2.15), (2.16), (2.17), (2.19) appear as integers, and these series are polynomials in x. In this case the logarithmic divergence at the front of the reflected wave (Equation (2.29)) disappears, because $\cos[\pi\nu/(2+s)] = 0$.

For $s \geqslant 2(\nu - 1)$ the number m can equal only one integer, zero, which occurs when $s = 2(\nu - 1)$. In this $w_2 = x$, $w_3 = E = 1$.

Substituting these relations into (2.6)-(2.8) we find the solution in simple form

$$u_2 = \frac{C}{2}r\left(1 + \nu\sqrt{\frac{\gamma p_0}{\omega}}\frac{t}{r^\nu}\right), \quad v_2 = \frac{C}{2}\nu\sqrt{\frac{\gamma p_0}{\omega}}\,r^{1-\nu}$$
$$p_2^\circ = -\frac{C}{2}\gamma p_0\nu, \quad u_3 = Cr, \quad v_3 = 0, \quad p_3^\circ = -C\gamma p_0\nu$$

These equations generalize the solution of the problem of reflection of a plane acoustic wave ($\nu = 1$).

The solution constructed satisfied the linear equations (2.1) also for $s < 2(\nu - 1)$. Thus in the case of a spherical wave in a homogeneous gas, that is for $s = 0$, $\nu = 3$, we obtain

$$w_2 = x(1-x), \qquad E = 0, \qquad w_3 = 0$$

The solution has the form

$$u_2 = \frac{C}{4}\left(1 - \frac{\gamma p_0}{\omega}\frac{t^2}{r^2}\right), \qquad v_2 = -\frac{C}{2}\frac{\gamma p_0}{\omega}\frac{t}{r^2}$$

$$p_2{}^\circ = -\frac{C}{2}\frac{\gamma p_0}{r} \qquad u_3 = 0, \qquad v_3 = 0, \qquad p_3{}^\circ = 0$$

A compression wave reflects as an expansion wave (and conversely), after which the undisturbed state is re-established.

The solution of the problem of a converging cylindrical acoustic wave in a homogeneous gas is obtained if we take $s = 0$, $\nu = 2$ in Equations (2.6)-(2.8), (2.14)-(2.19). This case was considered in [6] and [7]. In [6] the solution of the equation that we reduce to a hypergeometric one is found in the form of a series, and in [7] a converging cylindrical wave is constructed by superposition of plane waves. The expression for the pressure found by Zel'dovich [7] in the form of an integral depending on a parameter is obtained from (2.16) and (2.19) by use of the integral representation of the hypergeometric function [3]

$$F(\alpha, \beta; \theta; x) = \frac{\Gamma(\theta)}{\Gamma(\beta)\Gamma(\theta-\beta)}\int_0^1 z^{\beta-1}(1-z)^{\theta-\beta-1}(1-zx)^{-\alpha}\,dz \quad (\theta > \beta > 0)$$

However, for $s < 2(\nu - 1)$ we have $k < 0$, and the conditions of linearization of the equations are violated near $r = 0$. Thus the intensity of the converging wave becomes infinite according to the linear solution, which contradicts the assumption of applicability of Equation (2.12). Therefore, in the case $s < 2(\nu - 1)$ the linear solution under consideration is invalid near $r = 0$. In particular, the acoustic approximation for converging spherical and cylindrical waves in a uniform gas becomes invalid as the wave approaches the center (axis) of symmetry; there, the nonlinear solution constructed by Guderley [1] applies.

We note that the solution under consideration is asymptotically valid as $r \to 0$, $t \to 0$ also under assumptions somewhat more general than (1.1) regarding the undisturbed state. Thus the quantities ω and p_0 can be continuous functions of time which does not affect the process of reflection of the wave near $r = 0$, $t = 0$. The initial velocity of the particles need not be zero; if the estimates

$$v(r, 0) = o(r^{-1/2 s}) \quad \text{for } s < 2(\nu - 1),$$
$$v(r, 0) = o(r^{1/2 - 1/4 s - 1/2 \nu}) \quad \text{for } s \geqslant 2(\nu - 1)$$

hold for $r \to 0$, then it can be neglected in comparison with the velocity behind the shock, and the solution remains valid.

Thus if the density of an ideal gas near the center (axis, or plane) of symmetry is distributed according to a power-law, then the behavior of a converging shock may be different depending on the value of s. For $s < 2(\nu - 1)$ the shock-wave is intensified, or conserves a determined constant intensity near $r = 0$, whereas for $s > 2(\nu - 1)$ it weakens. The first case may be studied with the use of self-similar solutions, and the second on the basis of the linearized equations, where the linearization is correct for the converging wave.

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BIBLIOGRAPHY

1. Guderley, G., Starke kugelige und zylindrische Verdichtungsstösse in der Nähe des Kugelmittelpunktes bzw. der Zylinderachse. *Luftfahrtforschung* Vol. 19, No. 9, 1942.

2. Sedov, L.I., *Metody podobiia i razmernosti v mekhanike (Methods of Similarity and Dimensional Analysis in Mechanics)*. Gostekhteoretizdat, Moscow, 1957.

3. Ryzhik, I.M. and Gradshtein, I.S., *Tablitsy integralov, summ, riadov i proizvedenii (Tables of Integrals, Sums, Series and Products)*. Gostekhteoretizdat, Moscow-Leningrad, 1951.

4. Smirnov, V.I., *Kurs vysshei matematiki (Course in Higher Mathematics)*, Vol. 3, Chap. 2. Fizmatgiz, Moscow, 1958.

5. Whittaker, E.T. and Watson, G.N., *Kurs sovremennogo analiza (Modern Analysis)*, Chap. 2. Gostekhteoretizdat, Moscow-Leningrad, 1934.

6. Zababakhin, E.I. and Nechaev, M.H., Udarnye volny polia ikh kumuliatsiia (Shock-waves of a field and their cumulation). *ZhETF* Vol. 33, No. 2(8), 1957.

7. Zel'dovich, Ia.B., Tsilindricheskie avtomodel'nie akusticheskie volny (Cylindrical self-similar acoustic waves). *ZhETF* Vol. 33, No. 3(9), 1957.

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